Inversion Formulas for Multi-Dimensional Modified Stockwell Transforms

M. W. Wong[1]

Abstract We give a sample of results on the inversion formulas for multi-dimensional modified Stockwell transforms obtained by the author, his students and collaborators. All results can be found in the literature and complete references for them are given.

2010 Mathematics Subject Classification: 42C40, 47G10

Keywords: signal, Fourier transform, Gabor transform, wavelet transform, one-dimensional Stockwell transform, one-dimensional modified Stockwell transform, multi-dimensional modified Stockwell transform, inversion formula

1 One-Dimensional Modified Stockwell Transforms

A *signal* is a function f in $L^2(\mathbb{R})$. It is a function of time, which we denote by the variable x in $\mathbb{R}$. The usual *time-representation* of it is $f(x)$ at time x. An equally useful representation is by means of its *Fourier transform* $\hat{f}$, which we define as

$$\hat{f}(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx, \quad \xi \in \mathbb{R}.$$

The value $\hat{f}(\xi)$ is usually called the *frequency representation* or the *Fourier spectrum* of f at *frequency* ξ. The disadvantage of the Fourier transform in signal analysis lies in the fact that in order to calculate the Fourier spectrum of a signal f at a single frequency ξ, information about the signal f at almost every single time x is needed. One way to fix this is to introduce a window to concentrate on the duration of the signal for which we are interested in its spectrum. This is perhaps the pioneering idea due to Gabor [7] in signal

[1]Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario M3J 1P3, Canada. E-Mail: mwwong@mathstat.yorku.ca

analysis that we call *time-frequency analysis* in the modern era. To wit, let $\varphi \in L^2(\mathbb{R})$. Then we define the *Gabor transform* $G_\varphi f$ of a signal f with respect to the *window* φ by

$$(G_\varphi f)(b,\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi}\overline{\varphi(x-b)}f(x)\,dx, \quad b,\xi \in \mathbb{R}.$$

The quantity $(G_\varphi f)(b,\xi)$ is the time-frequency content of the signal f at time b and frequency ξ by placing a window φ at time b. One point to note, however, is that the window φ in the Gabor transform has a fixed size. It is much better to have a window adaptive to the frequency in the sense that the window is *narrow* for durations with high frequencies and *wide* for durations with low frequencies. That this can be done can be attributed to the arrival of the wavelet era.

Let $\varphi \in L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty.$$

This condition on φ is known as the *admissibility condition.* If a function φ in $L^2(\mathbb{R})$ satisfies the admissbility condition, then it is known as a *mother wavelet* and can serve as a window for the wavelet transform that we are going to recall. Let $\varphi \in L^2(\mathbb{R})$ be a mother wavelet. Then we define the *wavelet transform* $\Omega_\varphi f$ of a signal $f \in L^2(\mathbb{R})$ by

$$(\Omega_\varphi f)(b,\xi) = \int_{-\infty}^{\infty} f(x)\frac{1}{\sqrt{|a|}}\overline{\varphi\left(\frac{x-b}{a}\right)}dx$$

for all $b \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$. More details can be found in [4].

We can now introduce another time-frequency transform incorporating the principal features of the Gabor transform and the wavelet transform. Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1.$$

Then the *Stockwell transform* $S_\varphi f$ of a signal f with respect to the window φ is defined by

$$(S_\varphi f)(b,\xi) = (2\pi)^{-1/2}|\xi| \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\overline{\varphi(\xi(x-b))}\,dx$$

for all $b \in \mathbb{R}$ and $\xi \in \mathbb{R} \setminus \{0\}$. The first paper featuring the Stockwell transform is [20].

Some colleagues in the field of time-frequency analysis often identify the Stockwell transform with the Morlet wavelet transform in [4]. Maybe it is due to the following theorem.

Theorem 1.1 *For all* $f \in L^2(\mathbb{R})$,

$$(S_\varphi f)(b,\xi) = (2\pi)^{-1/2} e^{-ib\xi} \sqrt{|\xi|} (\Omega_\psi f)(b, 1/\xi), \quad b \in \mathbb{R}, \xi \in \mathbb{R} \setminus \{0\},$$

where

$$\psi(x) = e^{ix} \varphi(x), \quad x \in \mathbb{R}.$$

Notwithstanding the formula relating the Stockwell transform to the Morlet wavelet transform, the misleading similarities and the subtle differences between the Stockwell transforms and the Morlet wavelet transforms are explained on page 6 of [11].

For one-dimensional Stockwell transforms, the analysis, the applications and the computations can be found in, respectively, [5, 9, 10, 11, 21], [6, 8, 9, 10, 11, 19, 20, 22] and [1].

We can now introduce one-dimensional modified Stockwell transforms. Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = 1.$$

Then for $0 < s \leq \infty$, the modified Stockwell transform $S_{s,\varphi} f$ of a signal f is defined by

$$(S_{s,\varphi} f)(b,\xi) = (2\pi)^{-1/2} |\xi|^{1/s} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \overline{\varphi(\xi(x-b))}\, dx$$

for all $b \in \mathbb{R}$ and ξ in $\mathbb{R} \setminus \{0\}$.

In applications to imaging, as s increases from 1 to ∞, low frequencies are amplified and high frequencies diminished. To diminish low frequencies and amplify high frequencies, we look at $0 < s < 1$ instead of $1 \leq s \leq \infty$. Details with pictures can be found in [9, 11]. The main result on modified Stockwell transforms that we want to emphasize is the following inversion formula.

Theorem 1.2 *Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that*

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$$

and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty.$$

Then for all f and g in $L^2(\mathbb{R})$, we get

$$(f, g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S_{s,\varphi} f)(b, \xi) \overline{(S_{s,\varphi} g)(b, \xi)} \frac{db\, d\xi}{|\xi|^{1-(2/s')}},$$

where $\frac{1}{s} + \frac{1}{s'} = 1$ and

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi.$$

The aim of this paper is to introduce multi-dimensional modified Stockwell transforms and give inversion formulas for several classes of dilation matrix functions underpinning multi-dimensonal modified Stockwell transforms. All results in this paper have been published and the only contribution here is to present them in one place with relevant references [15, 16, 17]. Proofs are omitted.

2 Multi-Dimensional Modified Stockwell Transforms

Let $A \in \mathrm{GL}(n, \mathbb{R})$. Then for $0 < s \leq \infty$, the *multi-dimensional dilation operator* $D_{s,A}$ is defined by

$$(D_{s,A}\varphi)(x) = |\det A|^{-1/s} \varphi(A^{-1}x), \quad x \in \mathbb{R}^n.$$

If $s = 2$, then $D_{2,A} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary operator.

Let $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ be a mapping given by

$$\mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n, \mathbb{R})$$

and let $\varphi \in L^2(\mathbb{R}^n)$. Then for $0 < s \leq \infty$, we define the modified Stockwell transform $S_{s,A,\varphi}f$ of order s of a signal $f \in L^2(\mathbb{R}^n)$ by

$$(S_{s,A,\varphi}f)(b,\xi) = (2\pi)^{-n/2}|\det A_\xi|^{-1/s} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\overline{\varphi(A_\xi^{-1}(x-b))}dx$$

for all b and $\xi \in \mathbb{R}^n$. The function φ is known as the *window* of the modified Stockwell transform $S_{s,A,\varphi}$ of order s. We have the following simple relation of the modified Stockwell transform of order s with the modified Stockwell transform of order 2.

Proposition 2.1 *Let $A : \mathbb{R}^n \to \mathrm{GL}(n,\mathbb{R})$ be a mapping and let $\varphi \in L^2(\mathbb{R}^n)$ be a window. Then for $0 < s \leq \infty$,*

$$(S_{s,A,\varphi}f)(b,\xi) = |\det A_\xi|^{(1/2)-(1/s)}S_{2,A,\varphi}(b,\xi), \quad b,\xi \in \mathbb{R}^n,$$

for all $f \in L^2(\mathbb{R}^n)$.

A useful formula for the computations of modified Stockwell transforms is given in the following theorem.

Proposition 2.2 *Let $A : \mathbb{R}^n \to \mathrm{GL}(n,\mathbb{R})$ be a continuous mapping. Then for all windows $f \in L^2(\mathbb{R}^n)$,*

$$(S_{s,A,\varphi}f)(b,\xi) = |\det A_\xi^{-1}|^{1-(1/s)}e^{-ib\cdot\xi}f_{\xi,A_\xi}^{\vee}(b)$$

for all $b,\xi \in \mathbb{R}^n$, where $f_{\xi,A_\xi}^{\vee}$ is the inverse Fourier transform of f_{ξ,A_ξ} and

$$f_{\xi,A_\xi}(\zeta) = \hat{f}(\zeta)\overline{\hat{\varphi}(A_\xi^t(\zeta-\xi))}, \quad \zeta \in \mathbb{R}^n.$$

3 Moritoh Wavelet Transforms

Let $R : \mathbb{R}^n \to \mathrm{SO}(n,\mathbb{R})$. Then we define the *Moritoh wavelet transform* $W_{\frac{1}{|\xi|}R^{-1},\varphi}f$ of a signal f in $L^2(\mathbb{R}^n)$ with respect to the window φ in $L^2(\mathbb{R}^n)$ by

$$\left(W_{\frac{1}{|\xi|}R^{-1},\varphi}f\right)(b,\xi) = |\xi|^{n/2}\int_{\mathbb{R}^n} f(x)\overline{\varphi(|\xi|R_\xi(x-b))}dx$$

for all $b \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n \setminus \{0\}$. In fact,

$$\left(W_{\frac{1}{|\xi|}R^{-1},\varphi}f\right)(b,\xi) = \left(f, T_{-b}D_{2,\frac{1}{|\xi|}R_\xi^{-1}}\varphi\right)_{L^2(\mathbb{R}^n)}, \quad b \in \mathbb{R}^n, \xi \in \mathbb{R}^n \setminus \{0\},$$

where T_{-b} is the translation operator on $L^2(\mathbb{R}^n)$ given by

$$(T_{-b}g)(x) = g(x-b), \quad x \in \mathbb{R}^n,$$

for all $g \in L^2(\mathbb{R}^n)$. The Moritoh wavelet transform can be found in [14].

The following theorem gives the connection between the modified Stockwell transforms and the Moritoh wavelet transforms.

Theorem 3.1 *Let $R : \mathbb{R}^n \to \mathrm{SO}(n, \mathbb{R})$. For all $\xi \in \mathbb{R}^n \setminus \{0\}$, let*

$$A_\xi = \frac{1}{|\xi|} R_\xi^{-1}.$$

Let $\varphi \in L^2(\mathbb{R}^n)$. Then for $0 < s \leq \infty$,

$$(S_{s,A,\varphi}f)(b,\xi) = |\det A_\xi|^{(1/2)-(1/s)}(2\pi)^{-n/2}e^{-ib\cdot\xi}\left(W_{A,M_{(A_\xi^{-1})^t\xi}}f\right)(b,\xi)$$

for all $b \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n \setminus \{0\}$, where M_η, for every $\eta \in \mathbb{R}^n$, is the modulation operator on $L^2(\mathbb{R}^n)$ given by

$$(M_\eta g)(x) = e^{i\eta\cdot x}g(x), \quad x \in \mathbb{R}^n,$$

for all $g \in L^2(\mathbb{R}^n)$.

4 Constant Dilation Matrices

The first inversion formula for modified Stockwell transforms is provided by constant dilation matrices.

Theorem 4.1 *Let $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ be a constant matrix. Let φ be a nonzero function in $L^2(\mathbb{R}^n)$. Then for $0 < s \leq \infty$,*

$$(f,g)_{L^2(\mathbb{R}^n)} = \frac{1}{\|\varphi\|^2_{L^2(\mathbb{R}^n)}} \int_{\mathbb{R}^n} (S_{s,A,\varphi}f)(b,\xi)\overline{(S_{s,A,\varphi}g)(b,\xi)}|\det A|^{(2/s)-1}\,db\,d\xi$$

for all f and g in $L^2(\mathbb{R}^n)$.

Theorem 4.1 is in fact the inversion formula for multi-dimensional Gabor transforms. A proof of Theorem 4.1 requires only the Plancherel formula for the Fourier transform and Proposition 2.2.

5 Diagonal Matrix Dilations

We first give a lemma on diagonal matrix dilations.

Lemma 5.1 *Let $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ be given by*

$$(A_\xi^t)^{-1} = \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \xi_n \end{bmatrix}, \quad \xi \in \mathbb{R}^n.$$

Let $\varphi \in L^2(\mathbb{R}^n)$ and $\mathbf{1} = (1, 1, \ldots, 1)$. Then

$$\int_{\mathbb{R}^n} |\hat{\varphi}(A_\xi^t(\zeta - \xi))|^2 |\det A_\xi| \, d\xi = \int_{\mathbb{R}^n} |\hat{\varphi}(\eta - \mathbf{1})|^2 |\det A_\eta| \, d\eta.$$

A proof of Lemma 5.1 can be obtained by putting

$$\eta = A_\xi^t(\zeta - \xi)$$

and computing the Jacobian $\det\left(\frac{\partial \eta}{\partial \xi}\right)$.

We can now give the inversion formula for modified Stockwell transforms corresponding to diagonal matrix dilations.

Theorem 5.2 *Let $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ be such that*

$$A_\xi^t = \begin{bmatrix} \frac{1}{\xi_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\xi_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\xi_n} \end{bmatrix}$$

for all $\xi \in \mathbb{R}^n$ with $\xi_j \neq 0$, $j = 1, 2, \ldots, n$. Let $\varphi \in L^2(\mathbb{R}^n)$ be such that

$$c_\varphi = \int_{\mathbb{R}^n} |\hat{\varphi}(\eta - \mathbf{1})|^2 |\det A_\eta| \, d\eta < \infty.$$

Then for $0 < s \leq \infty$,

$$(f, g)_{L^2(\mathbb{R}^n)} = \frac{1}{c_\varphi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (S_{s,A,\varphi} f)(b, \xi) \overline{(S_{s,A,\varphi} g)(b, \xi)} \frac{db \, d\xi}{|\det A_\xi|^{1-(2/s)}}$$

for all f and g in $L^2(\mathbb{R}^n)$.

6 An Inversion Formula with Topological Obstruction

We begin with another inversion formula, provide some examples and illustrate the fact that the dimensions of the dilation matrices and hence the dimensions of the modified Stockwell transforms cannot be improved.

Theorem 6.1 *Let* $A : \mathbb{R}^n \to \mathrm{GL}(n,\mathbb{R})$ *be a continuous mapping such that*

$$\frac{1}{|\xi|} A_\xi^{-1} \in \mathrm{SO}(n,\mathbb{R}), \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Suppose that there exists a matrix $P \in \mathrm{O}(n,\mathbb{R})$ *such that*

$$A_\xi^{-1}\zeta = P A_\zeta^{-1}\xi, \quad \xi, \zeta \in \mathbb{R}^n.$$

Moreover, suppose that

$$A_\xi^t \xi = |\xi| e_1, \quad \xi \in \mathbb{R}^n,$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. *Let* $\varphi \in L^2(\mathbb{R}^n)$ *be such that*

$$c_\varphi = \int_{\mathbb{R}^n} |\hat{\varphi}(\eta - e_1)|^2 \frac{d\eta}{|\eta|^n} < \infty.$$

Then for all f *and* g *in* $L^2(\mathbb{R}^n)$,

$$(f, g)_{L^2(\mathbb{R}^n)} = \frac{1}{c_\varphi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (S_{s,A,\varphi} f)(b,\xi) \overline{(S_{s,A,\varphi} g)(b,\xi)} \frac{db\, d\xi}{|\xi|^{n((2/s)-1)}}.$$

We give some examples.

Example 6.2 The matrix-valued functions

$$A_\xi^{-1} = \begin{bmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{bmatrix}, \quad \xi \in \mathbb{R}^n,$$

$$A_\xi^{-1} = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ -\xi_2 & \xi_1 & \xi_4 & -\xi_3 \\ -\xi_3 & -\xi_4 & \xi_1 & -\xi_2 \\ -\xi_4 & -\xi_3 & -\xi_2 & \xi_1 \end{bmatrix}, \quad \xi \in \mathbb{R}^n,$$

and

$$A_\xi^{-1} = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 \\ -\xi_2 & \xi_1 & \xi_4 & -\xi_3 & \xi_6 & -\xi_5 & -\xi_8 & \xi_7 \\ -\xi_3 & -\xi_4 & \xi_1 & \xi_2 & \xi_7 & \xi_8 & \xi_5 & \xi_6 \\ -\xi_4 & \xi_3 & -\xi_2 & \xi_1 & \xi_8 & -\xi_7 & \xi_6 & \xi_5 \\ -\xi_5 & -\xi_6 & -\xi_7 & -\xi_8 & \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ -\xi_6 & \xi_5 & -\xi_8 & \xi_7 & -\xi_2 & \xi_1 & -\xi_4 & \xi_3 \\ -\xi_7 & \xi_8 & \xi_5 & -\xi_6 & -\xi_3 & \xi_4 & \xi_1 & -\xi_2 \\ -\xi_8 & -\xi_7 & \xi_6 & \xi_5 & -\xi_4 & -\xi_3 & \xi_2 & \xi_1 \end{bmatrix}, \quad \xi \in \mathbb{R}^n,$$

are dilation matrices satisfying the hypotheses of Theorem 6.1 when the dimension n is equal to, respectively, 2, 4 and 8.

Can we find examples for dimensions other than 2, 4 and 8? The answer is no. This is due to the fact by Bott and Milnor [2] to the effect that if $n \in \mathbb{N} \setminus \{1, 2, 4, 8\}$, then there are no continuous mappings $A : \mathbb{S}^{n-1} \to \mathrm{GL}(n, \mathbb{R})$ such that for every $\xi \in \mathbb{S}^{n-1}$, $A_\xi \xi$ is parallel to ξ.

7 Tensors and Inversion Formulas

A (1,2)-*tensor* F of *order* n is an $n \times n$ matrix of the form

$$F = \begin{bmatrix} F_{11}^1 & F_{21}^1 & \cdots & F_{n1}^1 \\ F_{12}^2 & F_{22}^2 & \cdots & F_{n2}^2 \\ \vdots & \vdots & \cdots & \vdots \\ F_{1n}^n & F_{2n}^n & \cdots & F_{nn}^n \end{bmatrix}.$$

A (1,1)-*tensor* G of *order* n is an $n \times n$ matrix of the form

$$G = [G_j^i]_{1 \le i,j \le n}.$$

The following lemma is due to Kalisa and Torrésani [12].

Lemma 7.1 *Let $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ be such that we can find be a $(1, 2)$-tensor F of order n and a $(1, 1)$-tensor G of order n such that*

$$(A_\xi^t)^{-1} = [F_{jl}^i \xi^l + G_j^i]_{1 \le i,j \le n}, \quad \xi = (\xi^1, \xi^2, \ldots, \xi^n) \in \mathbb{R}^n,$$

in the Einstein notation, i.e.,

$$\begin{aligned}(A_\xi^t)^{-1} &= \begin{bmatrix} \sum_{l=1}^n F_{1l}^1 \xi^l & \sum_{l=1}^n F_{2l}^1 \xi^l & \cdots & \sum_{l=1}^n F_{nl}^1 \xi^l \\ \sum_{l=1}^n F_{1l}^2 \xi^l & \sum_{l=1}^n F_{2l}^2 \xi^l & \cdots & \sum_{l=1}^n F_{nl}^2 \xi^l \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{l=1}^n F_{1l}^n \xi^l & \sum_{l=1}^n F_{2l}^n \xi^l & \cdots & \sum_{l=1}^n F_{nl}^n \xi^l \end{bmatrix} \\ &+ \begin{bmatrix} G_1^1 & G_2^1 & \cdots & G_n^1 \\ G_1^2 & G_2^2 & \cdots & G_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ G_1^n & G_2^n & \cdots & G_n^n \end{bmatrix}\end{aligned}$$

for all $\xi = (\xi^1, \xi^2, \ldots, \xi^n) \in \mathbb{R}^n$. For all $\zeta \in \mathbb{R}^n$, let $\eta_\zeta : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$\eta_\zeta(\xi) A_\xi^t(\zeta - \xi), \quad \xi \in \mathbb{R}^n. \tag{7.1}$$

Then for all $\varphi \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\hat{\varphi}(A_\xi^t(\zeta - \xi))|^2 |\det A_\xi| \, d\xi = \int_{\eta_\zeta(\mathbb{R}^n)} |\hat{\varphi}(\eta_\zeta)|^2 \frac{d\eta_\zeta}{|\det (I + F\eta_\zeta)|},$$

where ζ is a fixed but arbitrary element in $\mathbb{R}^n$.

The corresponding inversion formula is the following theorem.

Theorem 7.2 *Suppose that $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ is given by*

$$(A_\xi^t)^{-1} = [F_{jl}^i \xi^l + G_j^i]_{1 \le i,j \le n}$$

in the Einstein notation for all $\xi = (\xi^1, \xi^2, \ldots, \xi^n)$ in $\mathbb{R}^n$ where F is a $(1, 2)$-tensor of order n and G is a $(1, 1)$-tensor of order n. In addition, suppose that

$$\eta_\zeta(\mathbb{R}^n) = \mathbb{R}^n$$

for all $\zeta \in \mathbb{R}^n$, *where* η_ζ *is defined as in* (7.1). *Let* $\varphi \in L^2(\mathbb{R}^n)$ *be such that*

$$c_\varphi = \int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{|\det(I + F\xi)|} < \infty.$$

Then for $0 < s \leq \infty$,

$$(f, g)_{L^2(\mathbb{R}^n)} = \frac{1}{c_\varphi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (S_{s,A,\varphi} f)(b, \xi) \overline{(S_{s,A,\varphi} g)(b, \xi)} \, db \, d\xi$$

for all f *and* g *in* $L^2(\mathbb{R}^n)$.

8 Conclusions

Inversion formulas for multi-dimensional modified Stockwell transforms under appropriate admissibility conditions are the main results in this paper. Are there other inversion formulas for multi-dimensional Stockwell transforms as defined in this paper or in some other ways?

Acknowledgment: This research has been supported by the Natural Sciences and Engineering Research Council of Canada under Discovery Grant 0008562. I am grateful to Professor Ryuichi Ashino of Osaka Kyoiku University for inviting me to speak at the Research Institute for Mathematical Sciences of Kyoto University in the workshop "Image Analysis and Multi-dimensional Wavelet Analysis", which is a joint venture with the Institute of Mathematics for Industry.

References

[1] U. Battisti and L. Riba, Window-dependent bases for efficient representations of the Stockwell transform, *Appl. Comput. Harmon. Anal.* **40** (2016), 291–320.

[2] R. Bott and J. W. Milnor, On the parallelizability of the spheres, *Bull. Amer. Math. Soc.* **64** (1958), 87–89.

[3] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, 1995.

[4] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.

[5] J. Du, M. W. Wong and H. Zhu, Continuous and discrete inversion formulas for the Stockwell transform, *Integral Transforms Spec. Funct.* **18** (2007), 537–543.

[6] M. G. Eramian, R. A. Schincariol, L. Mansinha and R. G. Stockwell, Generation of auifer heterogeneity maps using two dimensional spectral texture segmentation techniques, *Math. Geology* **31** (1999), 327–348.

[7] D. Gabor, Theory of communication, *J. Inst. Elec. Eng. (London)* **93** (1946), 429–457.

[8] B. G. Goodyear, H. Zhu, R. A. Brown, J. R. Mitchell, Removal of phase artifacts from fMRI data using a Stockwell transform filter improves brain activity detection, *Magn. Reson. Med.* **51** (2004), 16–21.

[9] Q. Guo, S. Molahajloo and M. W. Wong, Modified Stockwell transforms and time-frequency analysis, in *New Developments in Pseudo-Differential Operators*, Editors: L. Rodino and M. W. Wong, Operator Theory: Advances and Applications **189**, Birkhäuser, 2009, 275–285.

[10] Q. Guo, S. Molahajloo and M. W. Wong, Phases of modified Stockwell transforms and instantaneous frequencies, *J. Math. Phys.* 2010; 51:052101, 11pp.

[11] Q. Guo and M. W. Wong, Modified Stockwell transforms, *Memorie della Accademia delle Scienze di Torino, Classe di Scienze, Fische. Matematiche e Naturali, Serie V,* **32** (2008), 3–20.

[12] C. Kalisa and B. Torrésani, n-dimensional affine Weyl–Heisenberg wavelets, *Ann. l' institut Henri Poincaré (A) Physique Théorique* **59** (1993), 201–236.

[13] Y. Liu and M. W. Wong, Inversion formulas for two-dimensional Stockwell transforms, in *Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis*, Editors: L. Rodino, B.-W. Schulze and M. W. Wong, Fields Institute Communications Series **52**, American Mathematical Society, 2007, 323–330.

[14] S. Moritoh, Wavelet transforms in Euclidean spaces - their relation with wave front sets and Besov, Triebel–Lizorkin spaces, *Tohoku Math. J.* **47** (1995), 555–565.

[15] L. Riba, *Multi-Dimensional Stockwell Transforms and Applications*, Ph.D. Dissertation, Università degli Studi di Torino, 2014.

[16] L. Riba and M. W. Wong, Continuous inversion formulas for multi-dimensional Stockwell transforms, *Math. Model. Nat. Phenom.* **8** (2013), 215–229.

[17] L. Riba and M. W. Wong, Continuous inversion formulas for multi-dimensional modified Stockwell transforms, *Integral Transforms Spec. Funct.* **26** (2015), 9–19.

[18] R. G. Stockwell, *S-Transform Analysis of Gravity Wave Activity from a Small Scale Network of Airglow Imagers*, Ph.D. Thesis, University of Western Ontario, 1999.

[19] R. G. Stockwell, Why use the S transform?, in *Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis*, Editors: L. Rodino, B.-W. Schulze and M. W. Wong, Fields Institute Communications Series **52**, American Mathematical Society, 2007, 279–309.

[20] R. G. Stockwell, L. Mansinha and R. P. Lowe, Localization of the complex spectrum: the S transform, *IEEE Trans. Signal Prccessing* **44** (1996), 998–1001.

[21] M. W. Wong and H. Zhu, A characterization of the Stockwell spectrum, in *Modern Trends in Pseudo-Differential Operators,* Editors: J. Toft, M. W. Wong and H. Zhu, Operator Theory: Advances and Applications **172**, Birkhauser, 2007, 251–257.

[22] H. Zhu, B. G. Goodyear, M. L. Lauzon, R. A. Brown, G. S. Mayer, L. Mansinha, A. G. Law and J. R. Mitchell, A new multiscale Fourier analysis for MRI, *Med. Phys.* **30** (2003), 1134–1141.